# Critical Behavior in Black Hole Thermodynamics

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In this talk I introduce the critical behavior occurring at the extremal limit of black holes. The extremal limit of black holes is a critical point and a phase transition takes place from the extremal black holes to their nonextremal counterparts. Some critical exponents satisfying the scaling laws are obtained. From the scaling laws we introduce the concept of the effective dimension of black holes and discuss the relationship between the critical behavior and the statistical interpretation of black hole entropy.

PACS numbers: 04.70.Dy, 04.60.Kz, 05.70.Jk,11.25.-w



## I. INTRODUCTION

Due to the celebrated works of Hawking [1] and Bekenstein [2], a black hole has a temperature proportional to the surface gravity on the black hole horizon and an entropy proportional to the area of event horizon. This put the first law of black hole thermodynamics on a solid fundament [3]. Also this made people believe that a black hole is a thermodynamic system.

The phase transition is an important phenomenon in the ordinary thermodynamics. It is therefore natural to ask whether there is any phase transition in the black hole thermodynamics. Davies [4] is the first person who discusses the critical behavior of Kerr-Newman black hole in terms of thermodynamics. For a Kerr-Newman black hole with mass $M$, charge $Q$ and angular momentum $J = aM$, the black hole entropy is

$$S = \frac{A}{4} = 2\pi\left(M^2 - \frac{1}{2}Q^2 + \sqrt{M^4 - J^2 - M^2Q^2}\right), \quad (1)$$

and the Hawking temperature is

$$T = \frac{\kappa}{2\pi} = \frac{\sqrt{M^2 - a^2 - Q^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}}. \quad (2)$$

According to the formula $C_{J,Q} = (\partial M/\partial T)_{J,Q}$, the heat capacity is

$$C_{J,Q} = \frac{MTS^3}{\pi J^2 + \pi Q^4/4 - T^2S^3}. \quad (3)$$

For the Schwarzschild black hole, $C = -M/T < 0$; For an extremal Kerr-Newman black hole, $C_{J,Q} \to 0^+$. Thus the heat capacity must diverge at certain points:

$$\begin{cases} Q_c = \sqrt{3}M/2, & \text{for RN black holes} \\ J_c = \sqrt{2\sqrt{3} - 3}M^2, & \text{for Kerr black holes} \end{cases} \quad (4)$$

Based on the infinity discontinuity of the heat capacity, Davies claimed that there are second-order phase transitions in black hole thermodynamics. Many authors have investigated the critical points in the different black holes [5–8]. Lousto [9] claimed that the critical points of Davies satisfy the fourth law of black hole thermodynamics: scaling laws, from which he assigned the effective spatial dimension of black holes to be two, and thought that his result is in complete agreement with the membrane picture of black holes [10]. However, he found later that his calculation has some errors. Thus his two-dimensional effective model of black holes becomes invalid.

On the other hand, some people think that there exist a critical point at the extremal limit of black holes and a second-order phase transition takes place from an extremal black hole to its nonextremal counterpart [11]. Kaburaki [12] found that the critical point at extremal limit also obeys the scaling laws by investigating the thermal equilibrium fluctuations of Kerr-Newman black hole in the micro-canonical ensemble. The present author and collaborators have investigated in detail the critical behavior in the BTZ black holes [13], dilaton black holes [14], and black $p$-branes [15], respectively, and obtained some interesting results. This article is devoted to showing these results in the nondilatonic black $p$-branes and arbitrary dimensional dilatonic black holes.

The organization of this paper is as follows. In next section we will discuss the critical behavior in the nondilatonic black $p$-branes and calculate the relevant critical exponents. In Sec. III we investigate the case of arbitrary dimensional dilatonic black holes. The conclusion and discussion are presented in Sec. IV.

## II. CRITICAL BEHAVIOR FOR NONDILATONIC BLACK $P$-BRANES

The nondilatonic black $p$-branes we will consider come from the following action [16]

$$S_{d+p} = \frac{1}{16\pi}\int d^{(d+p)}x\sqrt{-g}\left[R - \frac{2}{(d-2)!}F^2_{d-2}\right], \quad (5)$$

where $R$ is the scalar curvature and $F_{d-2}$ denotes the $(d-2)$-form anti symmetric tensor field. Performing the double-dimensional reduction by $p$ dimensions, one has the dilatonic $d$-dimensional action:

$$S_d = \frac{1}{16\pi}\int d^d\sqrt{-g}\left[R - 2(\nabla\phi)^2 - \frac{2}{(d-2)!}e^{-2a\phi}F^2_{d-2}\right], \quad (6)$$

where $\phi$ is the dilaton field, and the constant $a$ is

$$a = \frac{(d-3)\sqrt{2p}}{\sqrt{(d-2)(d+p-2)}}. \tag{7}$$

The magnetically charged black holes in the action (6) are

$$\begin{aligned} ds_d^2 &= \left[1-\left(\frac{r_+}{r}\right)^{d-3}\right]\left[1-\left(\frac{r_-}{r}\right)^{d-3}\right]^{1-(d-3)b} dt^2 \\ &+ \left[1-\left(\frac{r_+}{r}\right)^{d-3}\right]^{-1}\left[1-\left(\frac{r_-}{r}\right)^{d-3}\right]^{b-1} dr^2 \\ &+ r^2\left[1-\left(\frac{r_-}{r}\right)^{d-3}\right]^{b} d\Omega_{d-2}^2, \\ e^{a\phi} &= \left[1-\left(\frac{r_-}{r}\right)^{d-3}\right]^{-(d-3)b/2}, \\ F_{d-2} &= Q\varepsilon_{d-2}, \end{aligned} \tag{8}$$

where $\varepsilon_{d-2}$ is the volume form on the unit $(d-2)$-sphere, the constant $b$ is

$$b = 2p/(d-2)(p+1), \tag{9}$$

and the charge $Q$ is related to $r_\pm$ by

$$Q^2 = \frac{(d-3)(d+p-2)}{2(p+1)}(r_+r_-)^{d-3}. \tag{10}$$

Thus, one has the non-dilatonic black $p$-brane solutions in the action (5) [16]

$$ds_{d+p}^2 = e^{2m\phi}dy^i dy^i + e^{2n\phi}ds_d^2, \tag{11}$$

where $i = 1, 2, \cdots, p$, and

$$m = -\frac{\sqrt{2(d-2)}}{\sqrt{p(d+p-2)}}, \quad n = -\frac{mp}{d-2}. \tag{12}$$

The Hawking temperature and the Bekenstein-Hawking entropy per unit volume of $p$-branes for the black $p$-branes (11) are easily obtained:

$$T = \frac{d-3}{4\pi r_+}\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{1/(p+1)}, \tag{13}$$

$$S = \frac{\Omega_{d-2}}{4}r_+^{d-2}\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{p/(p+1)}, \tag{14}$$

where $\Omega_{d-2}$ is the volume of the unit $(d-2)$-sphere. The ADM mass per unit volume of $p$-branes is found to be

$$M = \frac{\Omega_{d-2}}{16\pi}\left[(d-2)r_+^{d-3} + \frac{d-2-p(d-4)}{p+1}r_-^{d-3}\right], \tag{15}$$

which satisfies the first law of thermodynamics,

$$dM = TdS + \Phi dQ, \tag{16}$$

where $\Phi = \Omega_{d-2}Q/[4\pi(d-3)r_+^{d-3}]$ is the chemical potential corresponding to the conservative charge $Q$. The heat capacity per unit volume of $p$-branes is

$$\begin{aligned} C_Q = &-\frac{\Omega_{d-2}r_+}{4}\frac{\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{p/(p+1)}}{\left[1-\frac{p+2d-5}{p+1}\left(\frac{r_-}{r_+}\right)^{d-3}\right]} \\ &\times\left[(d-2)r_+^{d-3} - \frac{d-2-p(d-4)}{p+1}r_-^{d-3}\right]. \end{aligned} \tag{17}$$

When the extremal limit ($r_- = r_+$) is approached, the temperature, entropy, and the heat capacity approach zero. When $1-(p+2d-5)(r_-/r_+)^{d-3}/(p+1) = 0$, the heat capacity diverges, which corresponds to the critical point of Davies in Kerr-Newman black holes [4].

In a self-gravitating thermodynamic system, the thermodynamical ensembles are not equivalent generally. To study the critical behavior in black hole thermodynamics, it is reasonable to choose the micro-canonical ensemble [12–15], in which the thermodynamical potential function is the entropy of system. Rewriting (16) we have

$$dS = \beta dM - \varphi dQ, \tag{18}$$

where $\beta = T^{-1}$ and $\varphi = \beta\Phi$. From (18) it follows that the intrinsic variables are $\{M, Q\}$ and the conjugate variables $\{\beta, -\varphi\}$.Thus the eigenvalues corresponding to the fluctuation modes $\beta$ and $\varphi$ are

$$\lambda_m = \left(\frac{\partial M}{\partial\beta}\right)_Q = -T^2C_Q, \tag{19}$$

$$\lambda_q = -\left(\frac{\partial Q}{\partial\varphi}\right)_M = -TK_M, \tag{20}$$

respectively, where

$$\begin{aligned} K_M &\equiv \beta\left(\frac{\partial Q}{\partial\varphi}\right)_M \\ &= \frac{4\pi(d-3)r_+^{d-3}}{\Omega_{d-2}}\left[1-\frac{d-2-p(d-4)}{(p+1)(d-2)}\left(\frac{r_-}{r_+}\right)^{d-3}\right] \\ &\quad\left\{\left[1+\frac{d-2-p(d-4)}{p+1)(d-2)}\left(\frac{r_-}{r_+}\right)^{d-3}\right]\right. \\ &+\frac{2}{(d-3)}\left(\frac{r_-}{r_+}\right)^{d-3}\left[\frac{d-3}{p+1}-\frac{d-2-p(d-4)}{(p+1)(d-2)}\right. \\ &\times\left.\left(1-\frac{p+d-2}{p+1}\left(\frac{r_-}{r_+}\right)^{d-3}\right)\right] \\ &\times\left.\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{-1}\right\}^{-1}. \end{aligned} \tag{21}$$

Obviously, the two eigenvalues approach zero as the extremal limit is approached. Hence some second moments

must diverge at the extremal limit [15]. As in the ordinary thermodynamics, the divergence of second moments means that the extremal limit is a critical point and a second-order phase transition takes place from the extremal to nonextremal black $p$-branes. As is well known, the extremal black $p$-braes are very different from the nonextremal in many aspects, such as the thermodynamic description and geometric structures [13]. In particular, it has been shown that some extremal black $p$-branes are supersymmetric and the supersymmetry is absent for the nonextremal black $p$-branes. So the occurrence of phase transition are consistent with the changes of symmetry. The extremal and nonextremal black $p$-branes are two different phases. The extremal black $p$-branes are in the disordered phase and the nonextremal black $p$-branes in the ordered phase. The order parameters of the phase transition can be defined as the differences of the conjugate variables between the two phases [12–15], such as $\eta_\varphi = \varphi_+ - \varphi_-$ can be regarded as the order parameters of black $p$-branes, where the suffixes "+" and "−" mean that the quantity is taken at the $r_+$ and $r_-$, respectively. The second-order derivatives of entropy with respect to the intrinsic variables are the inverse eigenvalues,

$$\zeta_m \equiv \left(\frac{\partial^2 S}{\partial M^2}\right)_Q = \lambda_m^{-1} = -\frac{\beta^2}{C_Q}, \tag{22}$$

$$\zeta_q \equiv \left(\frac{\partial^2 S}{\partial Q^2}\right)_M = \lambda_q^{-1} = -\frac{\beta}{K_M}. \tag{23}$$

Correspondingly, we can define the critical exponents of these quantities as follows,

$$\begin{aligned} \zeta_m &\sim \varepsilon_M^{-\alpha} \quad \text{(for Q fixed)}, \\ &\sim \varepsilon_Q^{-\psi} \quad \text{(for M fixed)}, \end{aligned} \tag{24}$$

$$\begin{aligned} \zeta_q &\sim \varepsilon_M^{-\gamma} \quad \text{(for Q fixed)}, \\ &\sim \varepsilon_Q^{-\sigma} \quad \text{(for M fixed)}, \end{aligned} \tag{25}$$

$$\begin{aligned} \eta_\varphi &\sim \varepsilon_M^{\beta} \quad \text{(for Q fixed)}, \\ &\sim \varepsilon_Q^{\delta^{-1}} \quad \text{(for M fixed)}, \end{aligned} \tag{26}$$

where $\varepsilon_M$ and $\varepsilon_Q$ represent the infinitesimal deviations of $M$ and $Q$ from their limit values. These critical exponents are found to be

$$\alpha = \psi = \gamma = \sigma = \frac{p+2}{p+1}, \quad \beta = \delta^{-1} = -\frac{1}{p+1}. \tag{27}$$

The critical exponents $\beta$ and $\delta^{-1}$ are negative, which shows the fact that the order parameter $\eta_\varphi$ diverges at the extremal limit. This is because the critical temperature is zero in this phase transition. It is easy to check that these critical exponents satisfy the scaling laws of the "first kind,"

$$\alpha + 2\beta + \gamma = 2, \quad \beta(\delta - 1) = \gamma, \quad \psi(\beta + \gamma) = \alpha. \tag{28}$$

That scaling laws (28) hold for the black $p$-branes is related to the fact that the black $p$-brane entropy (14) is a homogeneous function, satisfying

$$S(\lambda M, \lambda Q) = \lambda^{(d-2)/(d-3)} S(M, Q), \tag{29}$$

where $\lambda$ is a positive constant.

On the other hand, in an ordinary thermodynamic system, an important physical quantity related to phase transitions is the two-point correlation function, which has generally the form for a large distance,

$$G(r) \sim \frac{\exp(-r/\xi)}{r^{\bar{d}-2+\eta}}, \tag{30}$$

where $\eta$ is the Fisher's exponent, $\bar{d}$ is the effective spatial dimension of the system under consideration, and $\xi$ is the correlation length and diverges at the critical point. Similarly, the critical exponents of the correlation length for black $p$-branes can be defined as:

$$\begin{aligned} \xi &\sim \varepsilon_M^{-\nu} \quad \text{(for Q fixed)}, \\ &\sim \varepsilon_Q^{-\mu} \quad \text{(for M fixed)}. \end{aligned} \tag{31}$$

Combining with those in Eq. (28), these critical exponents form the scaling laws of the "second kind"

$$\nu(2-\eta) = \gamma, \quad \nu\bar{d} = 2 - \alpha, \quad \mu(\beta + \gamma) = \nu. \tag{32}$$

Because of the absence of quantum theory of gravity, we have not yet the correlation function of quantum black holes. Here we use the correlation function of scalar fields on the background of these black $p$-branes to mimic the one of black $p$-branes (from the obtained result below, it seems an appropriate approach to study the critical behavior of black holes at the present time). From the work of Traschen [17] who studied the behavior of scalar fields on the background of Reissner-Nordström black holes, it is found that the inverse surface gravity of the black hole plays the role of the correlation length of scalar fields. For the black $p$-branes, this conclusion holds as well. With the help of the surface gravity of black $p$-branes (13), we obtain

$$\nu = \mu = \frac{1}{p+1}. \tag{33}$$

Substituting (33) into (32), we find

$$\eta = -p, \quad \bar{d} = p. \tag{34}$$

When $p = 0$, the black $p$-branes (11) reduce to the non-dilatonic $d$-dimensional black holes. In this case, a similar calculation gives

$$\begin{aligned} &\alpha = \psi = \gamma = \sigma = 3/2, \quad \beta = \delta^{-1} = -1/2, \\ &\eta = -1, \quad \bar{d} = 1, \end{aligned} \tag{35}$$

from which it is easy to see that these critical exponents are independent of the dimensionality of spacetime and

parameters of black holes. These critical exponents are exactly the same as those of three dimensional BTZ black holes [13]. Recall the fact that the BTZ black holes are also exact non-dilatonic black hole solutions in string theory, we find that these critical exponents are universal for non-dilatonic black holes, an important feature of critical behavior in the non-dilatonic black holes. For the dilatonic black holes with the coupling constant $a$ obeying (7), we find that the effective spatial dimension is also $p$ (it is one for $p$=0). For a general $a$, the scaling laws still hold, but these critical exponents and effective dimension will depend on the coupling constant $a$ and the dimension $d$ of spacetime. In the next section, for the sake of generality, we will discuss case of an arbitrary dimensional dilatonic black holes.

## III. CRITICAL BEHAVIOR FOR ARBITRARY DIMENSIONAL DILATONIC BLACK HOLES

If the coupling constant $a$ between the dilaton and $(d-2)$-form is not related to the parameter $p$ in the action (6), instead of $b$ in (9), the parameter $b$ in the magnetically charged black hole solutions (8) should be

$$b = 2a^2(d-2)/[(d-3)(2(d-3)+a^2(d-2)]. \tag{36}$$

And the charge $Q$ becomes

$$Q^2 = \frac{(d-2)(d-3)^2}{2(d-3)+(d-2)a^2}(r_+r_-)^{d-3}. \tag{37}$$

The ADM mass of the solution is

$$M = \frac{\Omega_{d-2}(d-2)}{16\pi}\left[r_+^{d-3} + \frac{2(d-3)-(d-2)a^2}{2(d-3)+(d-2)a^2}r_-^{d-3}\right]. \tag{38}$$

The Hawking temperature and Bekenstein-Hawking entropy of the holes are

$$T = \frac{d-3}{4\pi r_+}\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{1-(d-2)b/2}, \tag{39}$$

$$S = \frac{\Omega_{d-2}r_+^{d-2}}{4}\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{(d-2)b/2}, \tag{40}$$

respectively. In this case, at the extremal limit ($r_- = r_+$), the entropy always vanishes because of $b>0$. But the behavior of Hawking temperature strongly depends on the parameter $a$. When $b=2/(d-2)$, the limiting temperature is finite. When $b>2/(d-2)$, the Hawking temperature diverges at the extremal limit. The Hawking temperature is zero as $b<2/(d-2)$. The behavior of thermodynamics is a general characteristic of dilatonic black hole thermodynamics.

Repeating the calculations in the previous section, we have the heat capacity

$$C_Q = -\frac{(d-2)\Omega_{d-2}r_+^{d-2}}{4}\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]^{(d-2)b/2} \times \frac{\left[1-\frac{2(d-3)-(d-2)a^2}{2(d-3)+(d-2)a^2}\left(\frac{r_-}{r_+}\right)^{d-3}\right]}{\left\{1-\left[2(d-3)+1-\frac{2(d-2)^2a^2}{2(d-3)+(d-2)a^2}\right]\left(\frac{r_-}{r_+}\right)^{d-3}\right\}}. \tag{41}$$

and $k_M$ in Eq. (21) is

$$\begin{aligned} K_M &\equiv \beta\left(\frac{\partial Q}{\partial\varphi}\right)_M \\ &= \frac{4\pi(d-3)}{\Omega_{d-2}}\frac{\left[1-\frac{2(d-3)-(d-2)a^2}{2(d-3)+(d-2)a^2}\left(\frac{r_-}{r_+}\right)^{d-3}\right]}{\left[1+\frac{2(d-3)-(d-2)a^2}{2(d-3)+(d-2)a^2}\left(\frac{r_-}{r_+}\right)^{d-3}\right]} \\ &\times\left\{\frac{1}{r^{d-3}}+\frac{2(d-3)^2-(d-2)a^2}{2(d-3)+(d-2)a^2}\right. \\ &\times\left.\frac{2r_-^{d-3}}{(d-3)r_+^{2(d-3)}}\left[1-\left(\frac{r_-}{r_+}\right)^{d-3}\right]\right\}. \end{aligned} \tag{42}$$

Thus we have the nonvanishing second moments in the micro-canonical ensemble,

$$\begin{aligned} &\langle\delta\beta\delta\beta\rangle = -k_B\frac{\beta^2}{C_Q}, \quad \langle\delta\varphi\delta\varphi\rangle = -k_B\frac{\beta}{K_M}, \\ &\langle\delta\beta\delta\Phi\rangle = k_B\frac{\beta\Phi}{C_Q}, \quad \langle\delta\Phi\delta\Phi\rangle = -k_B\left(\frac{T}{K_M}+\frac{\Phi^2}{C_Q}\right). \end{aligned} \tag{43}$$

From the above, we can see that, when $b=2/(d-2)$, these second moments are finite at the extremal limit. In this case, the extremal limit of black hole is not a critical point, just as the $a=1$ dilaton black holes in four dimensions [14]. Except this case, all second moments diverge as the extremal limit is approached. This means that the extremal limit is a critical point. Similar to the previous section, we can obtain the critical exponents in the " first kind"

$$\begin{aligned} &\alpha = \psi = \gamma = \sigma = \frac{4(d-3)^2+(d-4)(d-2)a^2}{(d-3)[2(d-3)+(d-2)a^2]}, \\ &\beta = \delta^{-1} = -\frac{2(d-3)^2-(d-2)a^2}{(d-3)[2(d-3)+(d-2)a^2]}, \end{aligned} \tag{44}$$

and in the "second kind"

$$\begin{aligned} &\nu = \mu = \frac{2(d-3)^2-(d-2)a^2}{(d-3)[2(d-3)+(d-2)a^2]}, \\ &\eta = -\frac{(d-2)^2a^2}{2(d-3)^2-(d-2)a^2}. \end{aligned} \tag{45}$$

From the scaling laws (32), we have the effective spatial dimension for an arbitrary dimensional dilatonic black hole

$$\bar{d} = \frac{(d-2)^2a^2}{2(d-3)^2-(d-2)a^2}. \tag{46}$$

In general, the effective dimension will be on longer an integer. In particular, when $a^2 > 2(d-3)^3/(d-2)$, the effective dimension is negative. The implication has been discussed in [14]. This can be explained in the intersecting M-brane configurations [19]. On the other hand, if $\bar{d}$ is remanded to be an integer, then the relation (46) gives us a constrain on the coupling constant $a$. To understand further its meaning is quite interesting.

## IV. CONCLUSION

From the above and combining with the results obtained in Refs. [13–15], we have the following conclusions:

(1) The extremal limit of dilatonic and non-dilatonic black $p$-branes is critical point and corresponding critical exponents obey the scaling laws.

(2) For the non-dilatonic black holes and black strings, the effective spatial dimension is one. This result is also reached in the BTZ black holes and 3-dimensional black strings [13].

(3) For the non-dilatonic black $p$-branes (black string for $p=1$), the effective dimension is $p$, so does it for the dilatonic black holes produced by the double-dimensional reduction of the non-dilatonic black $p$-branes.

(4) For other dilatonic black holes and black $p$-branes, the effective spatial dimension depends on the parameters in theories.

(5) Furthermore, near the extremal limit of the non-dilatonic black $p$-branes, from Eqs. (13)-(15), we have

$$S \sim T^p, \quad M - M_{\rm ext} \sim T^{p+1}, \tag{47}$$

where $M_{\rm ext}$ is the ADM mass of extremal black $p$-branes. Notice that the ADM mass and entropy of black $p$-branes are extensive quantities with respect to the volume of $p$-branes. Thus, near the extremal limit, the thermodynamic properties of non-dilatonic black $p$-branes can be described by the blackbody radiation in $(1+p)$ dimensions, which also further verify that the effective spatial dimension is $p$. For the dilatonic black holes (8) with constant $a$ satisfying (7), the equation (47) is also valid. Although the entropy of dilatonic black holes is not an extensive quantity, the entropy can be regarded as the entropy density of the non-dilatonic black $p$-branes. Thus it seems to imply that these dilatonic black hole entropy can also be explained as the way of $p$-branes, although the string coupling becomes very large in this case. Therefore, the entropy for the near extremal nondilatonic black holes, black strings, and black $p$-branes may be explained by free massless fields on the world volume.

Recall the recent progress in understanding entropy of black holes [18], in which the constant dilaton field seems to be a necessary condition. Therefore, our conclusions are in complete agreement with the result of these investigations. In addition, Klebanov and Tseytlin [20] found that there are nondilatonic black $p$-branes whose near-extremal entropy may be explained by free massless fields on the world volume. Thus we also give an interpretation why the Bekenstein-Hawking entropy may be given a simple world volume interpretation only for the non-dilatonic $p$-branes (including the non-dilatonic black holes).

This research was supported by the Center for Theoretical Physics of Seoul National University.

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